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On the nature of conjugate vortex flows

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Conjugate solutions of the equation of swirling cylindrical flow are considered in the case where the 'primary' flow has constant axial velocity. Numerical solutions are exhibited, which suggest a division of primary flows into four classes, and some conditions governing membership of these classes are derived. Finally, the case when the governing equation is slightly non-linear is considered.

1. Introduction

This paper is an attempt to answer certain questions in the theory of vortex breakdown as propounded by Benjamin (1962), and expanded by Fraenkel (1967). In his paper, Benjamin gives two examples of flows for which just one conjugate exists.† In one of these examples, the velocity on the axis is greater in the conjugate flow, and in the other the velocity on the axis is less in the conjugate flow. One purpose of the present paper is to explain why this should be, and to find out whether two conjugate flows can exist for the same primary flow. The other is to investigate the behaviour of the curve Γ, introduced by Fraenkel, from which important properties of conjugate flows can be derived.

Consider an inviscid fluid of unit density in steady axisymmetric swirling flow, in an infinite pipe of circular cross-section and radius $(2a)^{\frac{1}{2}}$. Let (r, θ, z) be cylindrical polar co-ordinates, (u, v, w) the corresponding velocity components, and put $y = \frac{1}{2}r^2$. Then the stream function ψ is defined by

$$\psi_y = w, \quad \psi_z = -ru,$$

and by $\psi = 0$ on $y = 0$.

Suppose that upstream a cylindrical flow A exists with velocity $(0, v_A(y), w_A(y))$, where $w_A(y) > 0$ on $[0, a]$. Then

$$\psi_A(y) = \int_0^y w_A(\tau)\, d\tau$$

is a monotonic function of y. Let $\psi_A(a) = b$, so that the volume flux through the pipe is $2\pi b$. Let the circulation be denoted by $\{8\pi^2 I_A(y)\}^{\frac{1}{2}}$ so that $I_A(y) = y v_A{}^2(y)$; and let the total head be denoted by $H_A(y)$, so that

$$H_A(y) = p_0 + \int_0^y v_A{}^2(\tau)/2\tau\, d\tau + \tfrac{1}{2}(v_A{}^2 + w_A{}^2),$$

where p_0 is the pressure on the axis.

† Although his equation (5.22) may have more than one solution for ξ in $[0, R]$, only one is physically relevant.

Functions $H(\psi)$ and $I(\psi)$ can then be defined parametrically for $0 \leqslant \psi \leqslant b$ by

$$H = H_A(\tau), \quad I = I_A(\tau) \quad \text{and} \quad \psi = \psi_A(\tau),$$

since $\psi_A(\tau)$ is a monotonic increasing function on $[0, a]$.

It is assumed that the flow A, known as the primary flow, is supercritical according to Benjamin's (1967) definition. This means that the flow A cannot support standing waves of small amplitude, or, mathematically, that the equation

$$\left.\begin{aligned} s_{yy} + \{\tfrac{1}{2}\gamma^2 y^{-1} + P(y)\} s &= 0, \\ \text{with} \qquad s(0) = s(a) &= 0, \end{aligned}\right\} \tag{1.1}$$

where
$$P(y) = -H''\{\psi_A(y)\} + \tfrac{1}{2}y^{-1}I''\{\psi_A(y)\},$$

has no negative eigenvalues γ^2.

Then the equation governing steady cylindrical flow may be written as

$$\psi_{yy} = H'(\psi) - \tfrac{1}{2}y^{-1}I'(\psi), \tag{1.2a}$$

with
$$\psi(0) = 0, \quad \psi(a) = b. \tag{1.2b,c}$$

For the derivation of equation (1.2) see Benjamin (1962, appendix §a).

The primary stream function $\psi_A(y)$ must be a solution of the boundary-value problem (1.2). There may be other solutions, the stream functions of the so-called conjugate flows. Not every solution, however, can represent a physically realizable flow. It is necessary that $\psi_y > 0$ everywhere in $(0, a)$, so that $y(\psi)$ is a single-valued function of ψ. This restriction also ensures that $0 \leqslant \psi \leqslant b$ for $0 \leqslant y \leqslant a$: this is necessary because the functions $H(\psi)$ and $I(\psi)$ are only defined for $0 \leqslant \psi \leqslant b$. Also, for the flow to be stable, it is necessary that $I_y > 0$. These conditions are applied to the flows discussed in §§2 and 3, but are abandoned in §4, which is primarily of mathematical interest. In §4, $H(\psi)$ and $I(\psi)$ will be extended analytically to values of ψ outside $[0, b]$.

Let the boundary condition (1.2c) be replaced by

$$\psi_y(0) = \lambda. \tag{1.3}$$

If the functions $w_A(\tau)$ and $v_A(\tau)$ are sufficiently smooth, they can be extended to all values of τ in such a way that the initial value problem (1.2a,b) and (1.3) has a unique solution $\psi(y, \lambda)$ on $[0, a]$ for all bounded values of λ (Fraenkel 1967, appendix).

For those values of λ for which

$$\psi(a, \lambda) = b, \tag{1.4}$$

$\psi(y, \lambda)$ is a solution of the boundary-value problem (1.2). Let the values of λ for which this is so be $\{\lambda_n\}$ where λ_0 is defined by $\psi_A(y) = \psi(y, \lambda_0)$, and $\ldots\lambda_{-1} < \lambda_0 < \lambda_1 \ldots$. Then, following Fraenkel, define

$$\xi(\lambda) = \psi(a, \lambda) - b \tag{1.5}$$

and
$$\eta(\lambda) = \psi_y(a, \lambda), \tag{1.6}$$

and let Γ be the curve defined parametrically in the (ξ, η)-plane by $\xi = \xi(\lambda)$,

$\eta = \eta(\lambda)$. Γ is simple and smooth, and intersects $\xi = 0$ at $\lambda = \lambda_n$. Also, if (1.2*a*) is linear (i.e. H and I are both quadratic functions of ψ), then (1.3) implies that $\psi(y, \lambda)$ is a linear function of λ, so that Γ is a straight line. It was shown by Fraenkel that the important properties of physical flows, viz. the change in flow force and whether or not the conjugate flow can support standing waves, are known when the form of Γ is known.

2. Computed solutions

The equations (1.2*a*, *b*) and (1.3) were integrated numerically by the Adams–Bashforth process on the Cambridge University Titan Computer for certain primary flows, and the curve Γ plotted in each case.

In §§ 2–4, the primary axial velocity is unity, and the radius of the pipe is $\sqrt{2}$, so that $\psi_A(y) = y$, and

$$w_A(y) = 1 \quad \text{for} \quad 0 \leqslant y \leqslant 1.$$

Then $\lambda_0 = 1$, and

$$I'(\psi) = 2\psi H'(\psi), \tag{2.1}$$

so that

$$H'(\psi) - \tfrac{1}{2}y^{-1}I'(\psi) = (1 - y^{-1}\psi)H'(\psi). \tag{2.2}$$

All the examples treated have the realistic feature

$$v_A/r \to \text{constant as } y \to 0;$$

i.e. the fluid has solid-body rotation on the axis of symmetry.

A convenient measure of how far a flow is supercritical or subcritical is Benjamin's parameter N, defined by

$$N = (c_+ + c_-)/(c_+ - c_-),$$

where $c_+(>0)$ and $c_-(\gtrless 0)$ are the propagation velocities in the main stream direction of very long waves propagating with and against the flow respectively. For a supercritical flow we have $c_- > 0$, $N > 1$, and for a subcritical flow we have $c_- < 0$, $N < 1$ (see Benjamin 1962).

Example (*a*)

The primary flow is given by

$$v_A(y) = \tfrac{1}{2}(9y + 6y^2)^{\frac{1}{2}} \quad \text{for} \quad 0 \leqslant y \leqslant 1. \tag{2.3}$$

Then

$$\psi_{yy} = H'(\psi) - \tfrac{1}{2}y^{-1}I'(\psi) = \tfrac{9}{4}(1 - y^{-1}\psi)(1 + \psi). \tag{2.4}$$

In this case a conjugate flow exists for $\lambda_1 = 2{\cdot}78$, with $\psi(y, \lambda_1) > \psi(y, 1)$ everywhere in (0, 1). There is no other conjugate flow. For the primary flow $N = 1{\cdot}10$ and for the conjugate flow $N = 0{\cdot}62$ (see figures 1 and 2).

Example (*b*)

The primary flow is given by

$$v_A(y) = (4y - 16y^2/15)^{\frac{1}{2}} \quad \text{for} \quad 0 \leqslant y \leqslant 1. \tag{2.5}$$

Then

$$\psi_{yy} = H'(\psi) - \tfrac{1}{2}y^{-1}I'(\psi) = 4(1 - y^{-1}\psi)(1 - \tfrac{2}{5}\psi). \tag{2.6}$$

In this case a conjugate flow exists for $\lambda_{-1} = 0{\cdot}08$, with $\psi(y, \lambda_{-1}) < \psi(y, 1)$ everywhere in $(0, 1)$. There is no other conjugate flow. For the primary flow $N = 1{\cdot}02$ and for the conjugate flow $N = 0{\cdot}52$ (see figures 3 and 4).

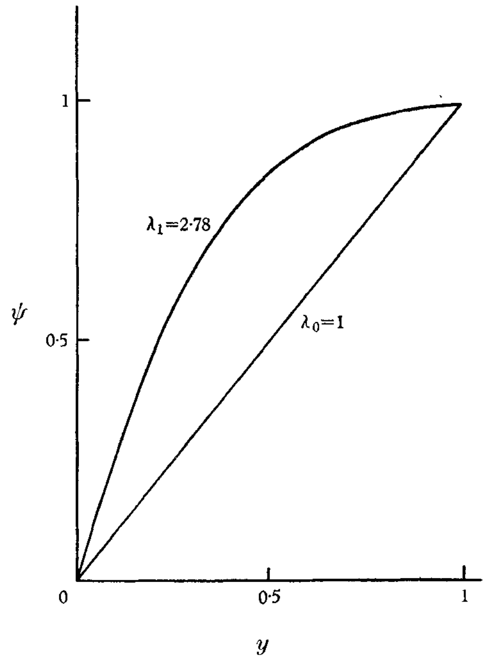

FIGURE 1. Conjugate solutions of equation (2.4).

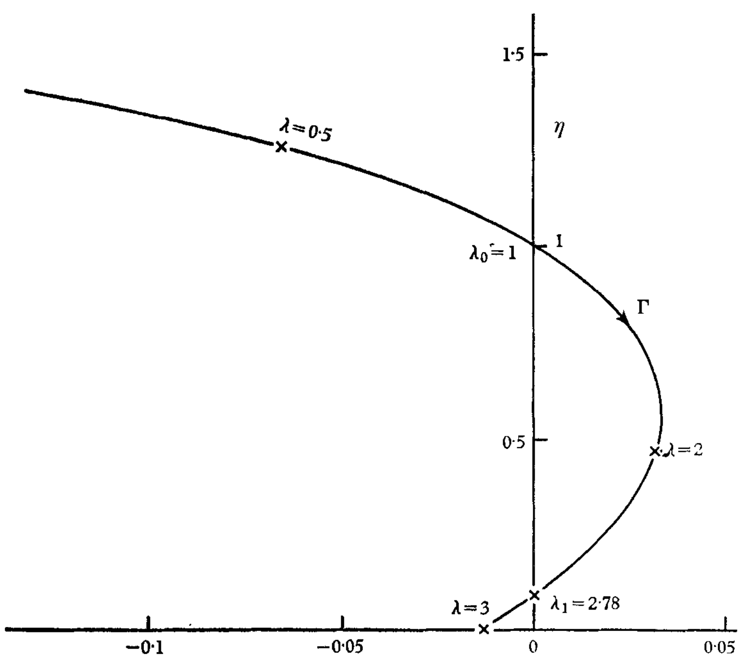

FIGURE 2. The curve Γ for the primary flow $v_A(y) = \frac{1}{2}(9y+6y^2)^{\frac{1}{2}}$, $w_A(y) = 1$ on $0 \leqslant y \leqslant 1$.

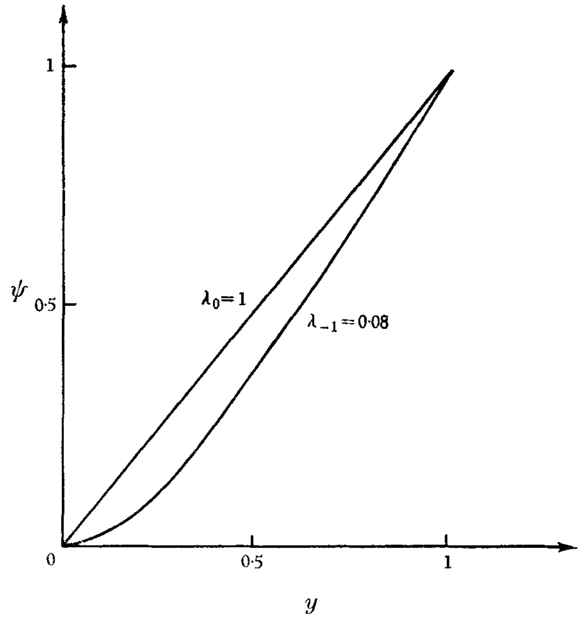

FIGURE 3. Conjugate solutions of equation (2.6).

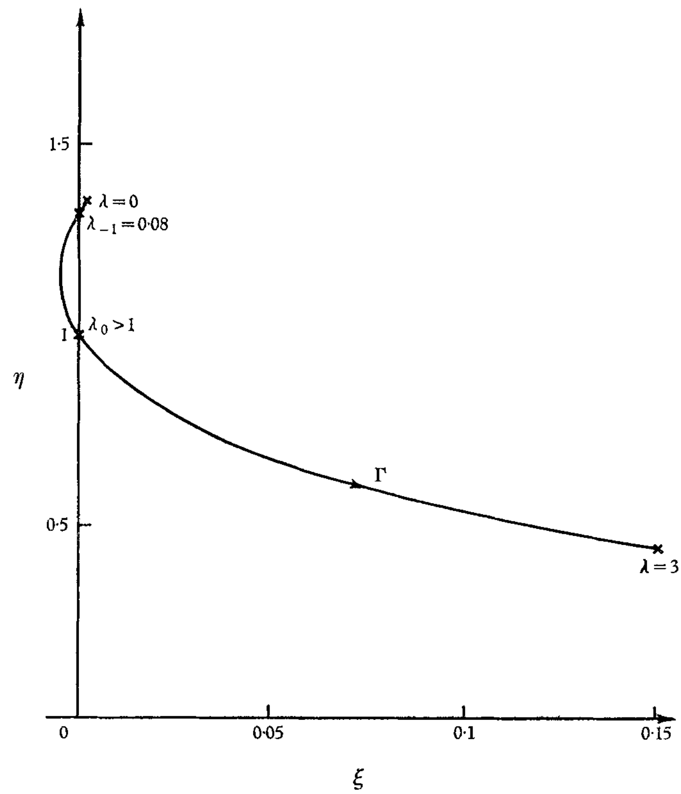

FIGURE 4. The curve Γ for the primary flow $v_A(y) = (4y-16y^2/15)^{\frac{1}{2}}$, $w_A(y) = 1$ on $0 \leqslant y \leqslant 1$.

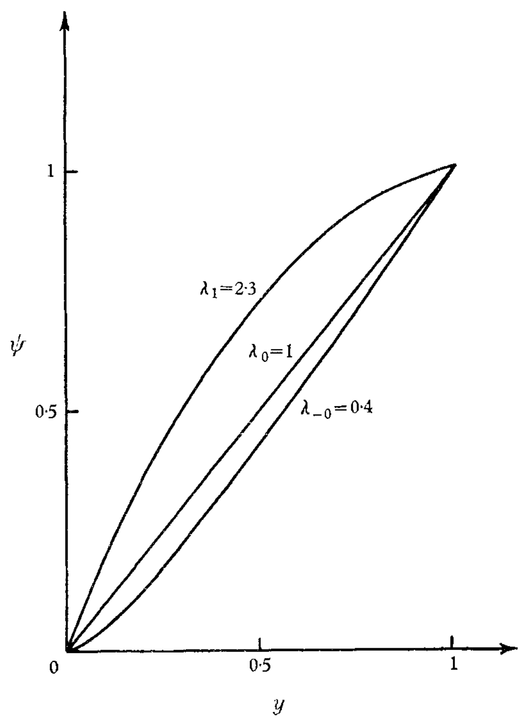

FIGURE 5. Conjugate solutions of equation (2.8).

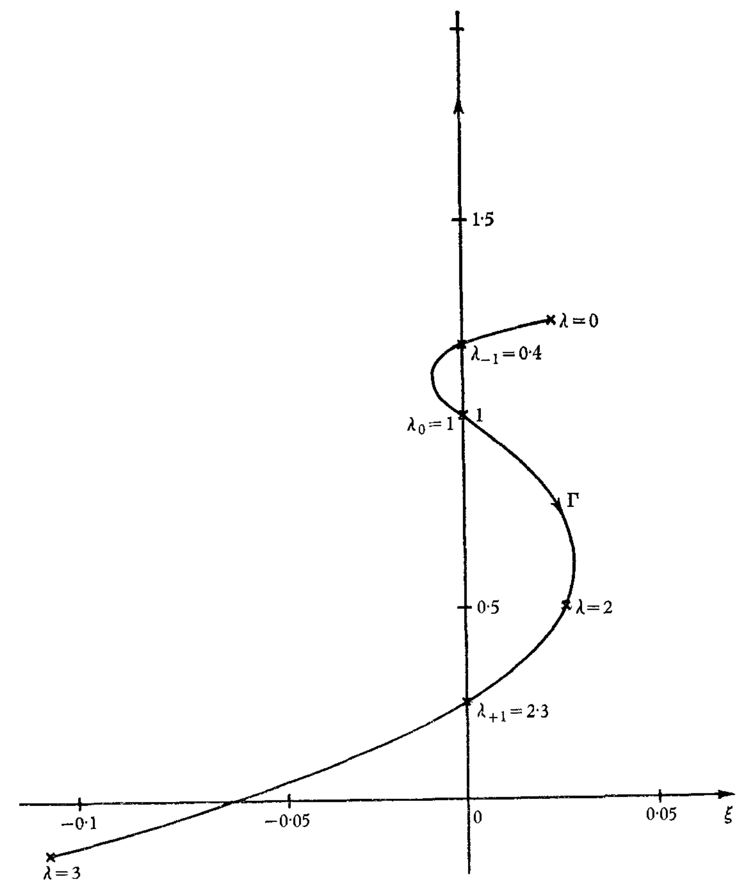

FIGURE 6. The curve Γ for the primary flow $v_A(y) = \frac{5}{4}(4y - 8y^2 + 7y^3)^{\frac{1}{2}}$, $w_A(y) = 1$ on $0 \leqslant y \leqslant 1$.

Example (*c*)

The primary flow is given by

$$v_A(y) = \tfrac{5}{4}(4y - 8y^2 + 7y^3)^{\frac{1}{2}} \quad \text{for} \quad 0 \leqslant y \leqslant 1. \tag{2.7}$$

Then

$$\psi_{yy} = H'(\psi) - \tfrac{1}{2}y^{-1}I'(\psi) = \tfrac{25}{4}(1 - y^{-1}\psi)(1 - 3\psi + \tfrac{7}{2}\psi^2). \tag{2.8}$$

In this case there are two conjugate flows: one for $\lambda_{-1} = 0{\cdot}4$ and one for $\lambda_1 = 2{\cdot}3$. For the primary flow $N = 1{\cdot}05$; for the conjugate flow with $\lambda = \lambda_{-1}$, $N = 0{\cdot}65$, and for the conjugate flow with $\lambda = \lambda_1$, $N = 0{\cdot}81$ (see figures 5 and 6).

3. The classification of conjugate flows

(i) The only conjugate flows relevant to the physical problem are those where $\lambda = \lambda_{-1}$ or $\lambda = \lambda_1$, for reasons given in Benjamin (1962, page 608). We may therefore divide primary flows into four classes:

Class O—where no conjugate flow exists.

Class I*a*—where λ_1, but not λ_{-1}, exists.

Class I*b*—where λ_{-1}, but not λ_1, exists.

Class II—where both λ_1 and λ_{-1} exist.

Example (*a*) above is of class I*a*, example (*b*) of class I*b*, and example (*c*) of class II. If equation (1.2*a*) is linear (i.e. if $H'(\psi)$ is constant), the flow will be of class O, except when the homogeneous linear equation

$$s_{yy} + y^{-1}H'(\psi)s = 0, \quad \text{with} \quad s(0) = s(1) = 0 \tag{3.1}$$

happens to have an eigensolution, in which case every λ corresponds to a conjugate flow, since an arbitrary multiple of such an eigensolution may be added to the primary stream function $\psi_A = y$.

(ii) We now derive some necessary conditions for the existence of conjugate flows. The partial derivative ψ_λ of the function $\psi(y, \lambda)$ defined by (1.2*a*, *b*) and (1.3) satisfies

$$\psi_{\lambda yy} + \{y^{-1}H'(\psi) - (1 - y^{-1}\psi)H''(\psi)\}\psi_\lambda = 0,$$

with

$$\psi_\lambda(0, \lambda) = 0, \quad \psi_{\lambda y}(0, \lambda) = 1;$$

so that for $\psi = \psi(y, \lambda_0) = \psi_A(y)$, if we define

$$(\lambda - 1)\psi_\lambda(y, \lambda_0) = \chi(y, \lambda), \tag{3.2}$$

χ satisfies

$$\chi_{yy} + y^{-1}H'(y)\chi = 0, \tag{3.3a}$$

with

$$\chi(0, \lambda) = 0, \quad \chi_y(0, \lambda) = \lambda - 1. \tag{3.3b}$$

It is now possible to compare $\chi(y, \lambda)$ with the function

$$\phi(y, \lambda) = \psi(y, \lambda) - y. \tag{3.4}$$

The function $\phi(y, \lambda)$ satisfies

$$\phi_{yy} + y^{-1}H'(\phi + y)\phi = 0, \tag{3.5a}$$

with $$\phi(0,\lambda) = 0, \quad \phi_y(0,\lambda) = \lambda - 1. \tag{3.5b}$$

If $\lambda = \lambda_n$, so that $\psi(y,\lambda_n)$ is the stream function of a conjugate flow, $\phi(y,\lambda_n)$ also satisfies $\phi(1,\lambda_n) = 0$.

First, we note that if the primary flow is supercritical, the function $\psi_\lambda(y,\lambda_0)$ has no zero in $(0,1]$ (see Benjamin 1962, §4.4; Fraenkel 1967, page 90), and hence $\chi(y,\lambda)$ has the same sign as $\lambda - 1$ on $(0,1]$.

From (3.3*a*) and (3.5*a*) we obtain

$$\chi\phi_{yy} - \phi\chi_{yy} = y^{-1}\{H'(y) - H'(\phi+y)\}\phi\chi.$$

Hence $$\chi^2\left(\frac{\phi}{\chi}\right)_y = \int_0^y \{H'(\tau) - H'(\phi+\tau)\}\phi\chi\frac{d\tau}{\tau}. \tag{3.6}$$

Also, from (3.3*b*) and (3.5*b*)

$$\lim_{y\to 0}\frac{\phi}{\chi} = \lim_{y\to 0}\frac{\phi_y}{\chi_y} = 1. \tag{3.7}$$

(*a*) Suppose that $H''(\psi) < 0$ on $[0,1]$ and that $\lambda > 1$. Then both χ and $\{H'(\tau) - H'(\phi+\tau)\}\phi$ are positive on $(0,1]$, so that the integral in (3.6) is positive. Hence ϕ/χ is an increasing function, and from (3.7) we see that $\phi > \chi$ on $(0,1]$. Since $\chi > 0$, we must have $\phi(1,\lambda) > 0$, and a conjugate solution is impossible.

(*b*) Suppose that $H''(\psi) > 0$ on $[0,1]$ and that $\lambda < 1$. Then both χ and $\{H'(\tau) - H'(\phi+\tau)\}\phi$ are negative on $(0,1]$, so that the integral in (3.6) is positive. Hence ϕ/χ is an increasing function, and from (3.7) we see that $\phi < \chi < 0$ on $(0,1]$. We must have $\phi(1,\lambda) < 0$, and a conjugate solution is impossible.

Therefore we conclude: *if $H''(\psi) < 0$ on $[0,1]$, no conjugate solution is possible with $\lambda > \lambda_0$. If $H''(\psi) > 0$ on $[0,1]$, no conjugate solution is possible with $\lambda < \lambda_0$. For a primary flow to belong to class II, as defined above, $H''(\psi)$ must change sign somewhere in $(0,1)$.*

Let us verify that our previous results are consistent with this conclusion. In example (*a*) of §2, $$H''(\psi) = \tfrac{9}{4} > 0,$$

so that no conjugate flow with $\lambda < \lambda_0$ can exist. In example (*b*),

$$H''(\psi) = -\tfrac{8}{5} < 0,$$

so that no conjugate flow with $\lambda > \lambda_0$ can exist. In example (*c*),

$$H''(\psi) = \tfrac{25}{4}(-3+7\psi),$$

which changes sign at $\psi = \frac{3}{7}$, so that conjugate solutions may exist on both sides of the primary flow.

In Benjamin's (1962) example 1,

$$H''(\psi) = \kappa^2 > 0,$$

and the flow is of class I*a*.

(iii) It appears that the function $H''(\psi)$ is important in determining the nature of conjugate flows. When $H''(\psi)$ is positive, conjugate flows in $\psi \geqslant y$ are in some sense encouraged, and those in $\psi \leqslant y$ discouraged, and vice versa when $H''(\psi)$ is negative. The assumption that the primary flow is supercritical puts an upper bound on the infimum of $H'(y)$ on $[0,1]$ [see (3.3a)]; in fact if $H'(y) > \frac{1}{4}j_{1,1}^2 = 3{\cdot}67$ on $[0,1]$ the flow cannot be supercritical. It can be shown that the parameter N for the primary flow is simply related to $H'(y)$.† In fact, N^2 is the smallest positive eigenvalue of the equation

$$s_{yy} + N^2 y^{-1} H'\{\psi_A(y)\} s = 0,$$

with

$$s(0) = s(1) = 0.$$

According to Benjamin (1965), in all flows where vortex breakdown has been observed, the conjugate flow has had a smaller velocity on the axis than the primary flow; i.e. the flow has been of class Ib. This is to be expected, as for real flows in pipes it is usually true that $v_{rr} < 0$ and v/r decreases outwards, so that $rv_r < v$, and hence

$$H''(\psi) = \frac{v_r^2}{r^2} + \frac{vv_{rr}}{r^2} + \frac{vv_r}{r^3} - \frac{2v^2}{r^4} < 0.$$

4. Conjugate flows when the governing equation is slightly non-linear

If

$$v_A(y) = \sigma y^{\frac{1}{2}}, \quad w_A(y) = 1, \tag{4.1}$$

equation (1.2a) becomes

$$\psi_{yy} = \sigma^2(1 - y^{-1}\psi), \tag{4.2}$$

which is linear. The general solution is

$$\psi = y + Ay^{\frac{1}{2}}J_1(2\sigma y^{\frac{1}{2}}) + By^{\frac{1}{2}}Y_1(2\sigma y^{\frac{1}{2}}).$$

Solutions satisfying $\psi(0) = 0$ and $\psi(1) = 1$ exist only if $J_1(2\sigma) = 0$, so that $\sigma = \frac{1}{2}j_{1,n}$ where $j_{1,n}$ is the nth zero of J_1. Moreover, the primary flow is subcritical when $|\sigma| > \frac{1}{2}j_{1,1}$ and supercritical when $|\sigma| < \frac{1}{2}j_{1,1}$; when $|\sigma| = \frac{1}{2}j_{1,1}$ it is just critical.

If $\sigma = \frac{1}{2}j_{1,1}$,

$$\psi(y, \lambda) = y + (\lambda - 1)\sigma^{-1}y^{\frac{1}{2}}J_1(2\sigma y^{\frac{1}{2}}), \tag{4.3}$$

and Γ consists of the η-axis $\xi = 0$.

Figure 7 shows the curve Γ corresponding to a primary flow

$$\left.\begin{aligned} v_A(y) &= \sigma y^{\frac{1}{2}}(1-y), \\ w_A(y) &= 1, \end{aligned}\right\} \tag{4.4}$$

where $\sigma = \frac{1}{2}j_{1,1}$. In this case Γ is a double spiral. (The curve in figure 6 is part of such a spiral.) This raises the question: given a family of functions

$$f(y, \psi, \epsilon) \equiv H'(\psi, \epsilon) - \tfrac{1}{2}y^{-1}I'(\psi, \epsilon),$$

such that for $\epsilon = 0$ the problem (1.2) is linear, but still has conjugate solutions, how does Γ, which is (say) a spiral for $\epsilon \neq 0$, reduce to the η-axis as $\epsilon \to 0$?

Suppose that the primary flow is

$$\left.\begin{aligned} v_A(y) &= \sigma y^{\frac{1}{2}} + \epsilon v_1(y), \\ w_A(y) &= 1, \end{aligned}\right\} \tag{4.5}$$

† Put $W = 1 - c$ and $\alpha = 0$ into Benjamin's (1962) equation (A 24), noting that c_+ and c_- are the roots of $N^2 = (1-c)^{-2}$.

where ϵ is a small parameter. Then

$$I'(\psi) = 2\sigma^2\psi + 2\epsilon\psi g(\epsilon, \psi),$$

say, so that (1.2*a*) becomes

$$\psi_{yy} = (1 - y^{-1}\psi)(\sigma^2 + \epsilon g(\epsilon, \psi)), \tag{4.6}$$

with $$\psi(0) = 0, \quad \psi_y(0) = \lambda.$$

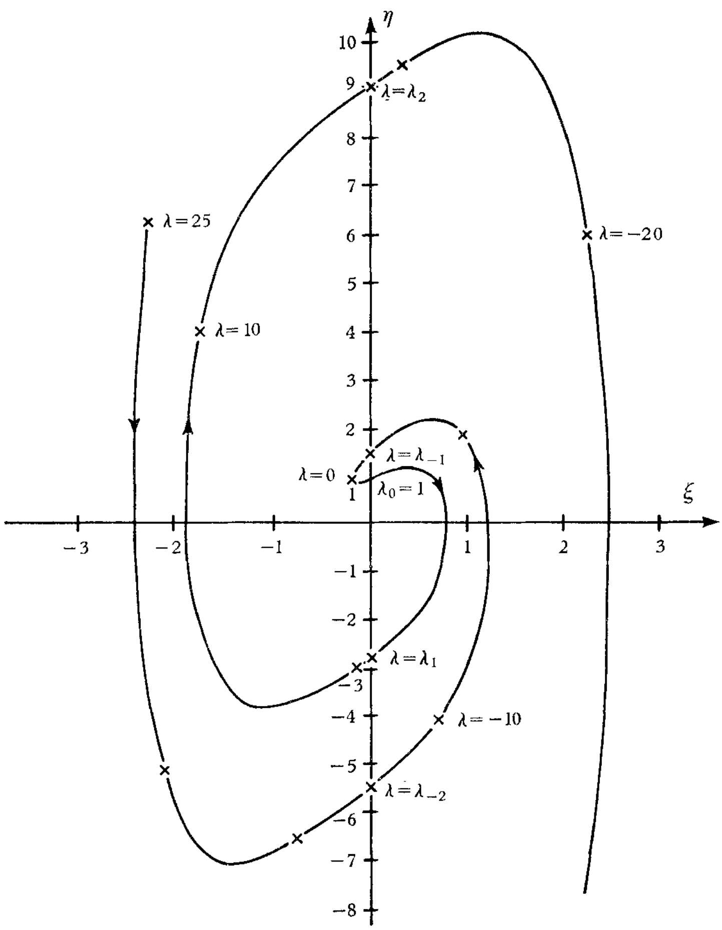

FIGURE 7. The curve Γ for the primary flow $v_A(y) = \sigma y^{\frac{1}{2}}(1-y)$, $w_A(y) = 1$.

Suppose that $\psi(y, \lambda)$ can be expanded as a power series in ϵ:

$$\psi(y, \lambda) = \sum_{n=0}^{\infty} \epsilon^n \psi_n(y, \lambda). \tag{4.7}$$

Substitute (4.7) into (4.6), putting

$$\epsilon g\left(\epsilon, \sum_{n=0}^{\infty} \epsilon^n \psi_n\right) = \sum_{n=1}^{\infty} \epsilon^n g_n(\psi_0, \psi_1, \ldots, \psi_{n-1})$$

(note that g_n is independent of ψ_n). Then, equating powers of ϵ, we obtain

$$\left.\begin{aligned} &\psi_{0yy} = \sigma^2(1-y^{-1}\psi_0), \\ \text{with} \quad &\psi_0(0,\lambda) = 0, \quad \psi_{0y}(0,\lambda) = \lambda; \end{aligned}\right\} \tag{4.8}$$

$$\left.\begin{aligned} \text{and} \quad &\psi_{nyy} + \sigma^2 y^{-1}\psi_n = f_n(y) \quad \text{for} \quad n > 0 \\ \text{with} \quad &\psi_n(0,\lambda) = 0, \quad \psi_{ny}(0,\lambda) = \lambda, \end{aligned}\right\} \tag{4.9}$$

where
$$f_n \equiv g_n(\psi_0, \psi_1, \ldots, \psi_{n-1}) - \sum_{r=0}^{n-1} \frac{\psi_r g_{n-r}}{y}. \tag{4.10}$$

Therefore ψ_0 is the unperturbed solution (4.3), and equation (4.9), which is linear, may be easily solved by the method of variation of parameters. In fact, the solution of (4.9) is

$$\psi_n = A_n(y) y^{\frac{1}{2}} J_1(2\sigma y^{\frac{1}{2}}) + B_n(y) y^{\frac{1}{2}} Y_1(2\sigma y^{\frac{1}{2}}), \tag{4.11}$$

where
$$A_n(y) = \int_0^y -\pi F_n(\tau) \tau^{\frac{1}{2}} Y_1(2\sigma\tau^{\frac{1}{2}})\, d\tau,$$

$$B_n(y) = \int_0^y \pi F_n(\tau) \tau^{\frac{1}{2}} J_1(2\sigma\tau^{\frac{1}{2}})\, d\tau,$$

and
$$F_n(\tau) \equiv f_n\{\psi_0(\tau), \psi_1(\tau), \ldots, \psi_{n-1}(\tau)\}.$$

In this way $\psi_0, \psi_1, \ldots$, can be found successively.

We are primarily interested in the dependence of ψ on λ at the point $y = 1$. This depends on the form of the function g. However, it can be said that if g_1 is a polynomial of degree m in ψ_0, ψ_n will be a polynomial of degree at least $mn+1$ in λ; so that the power series (4.7) for ψ is likely to fail when $\lambda^{-m} = O(\epsilon)$. It is then necessary to find some other way of attacking the problem (4.6). In general the form of the power series (4.7) will suggest a change of variables, which, when applied to the initial value problem (4.6), will allow an expansion in powers of ϵ remaining valid for unbounded λ. This rescaling will depend on the form of $v_1(y)$ in (4.5), and can be illustrated only for particular cases.

Example: primary flow given by

$$\left.\begin{aligned} v_A(y) &= \sigma y^{\frac{1}{2}}(1+\tfrac{2}{3}\epsilon y)^{\frac{1}{2}}, \\ w_A(y) &= 1. \end{aligned}\right\} \tag{4.12}$$

When $\epsilon = 1$ and $\sigma = \frac{3}{2}$ this reduces to the primary flow given by (2.3) above.

Now, (4.12) leads to the equation

$$\left.\begin{aligned} &\psi_{yy} = \sigma^2(1-\psi y^{-1})(1+\epsilon\psi), \\ \text{with} \quad &\psi(0) = 0, \quad \psi_y(0) = \lambda. \end{aligned}\right\} \tag{4.13}$$

Here $g(\epsilon, \psi) = \sigma^2\psi$, so that

$$g_n(\psi) = \sigma^2\psi_{n-1} \quad \text{for} \quad n \geqslant 1.$$

From (4.3),
$$\psi_0 = y + (\lambda-1)\sigma^{-1} y^{\frac{1}{2}} J_1(2\sigma y^{\frac{1}{2}}),$$

and hence
$$F_1(y) = -\sigma(\lambda-1) J_1(2\sigma y^{\frac{1}{2}})\{y^{\frac{1}{2}} + (\lambda-1)\sigma^{-1} J_1(2\sigma y^{\frac{1}{2}})\}. \tag{4.14}$$

Therefore A_1, B_1 and ψ_1 are quadratic in λ, and in general ψ_n is a polynomial of degree $n+1$ in λ, whose coefficients are easily found by integration.

When $\sigma = \frac{1}{2}j_{1,1} = 1{\cdot}916$, so that the curve Γ for the unperturbed flow reduces to the η-axis, we find that

$$\left.\begin{aligned} A_1 &= 0{\cdot}1325(\lambda-1)+0{\cdot}0028(\lambda-1)^2 \\ \text{and}\qquad B_1 &= -0{\cdot}3254(\lambda-1)-0{\cdot}1339(\lambda-1)^2. \end{aligned}\right\} \tag{4.15}$$

Then, to first order in ϵ,

$$\left.\begin{aligned} \xi &= \{-0{\cdot}1342(\lambda-1)-0{\cdot}0552(\lambda-1)^2\}\epsilon \\ \text{and}\qquad \eta &= 1-0{\cdot}4028(\lambda-1)+\{-0{\cdot}1339(\lambda-1)-0{\cdot}0152(\lambda-1)^2\}\epsilon. \end{aligned}\right\} \tag{4.16}$$

Therefore, when ϵ is sufficiently small, that portion of Γ corresponding to bounded values of $\lambda-1$ can be replaced by a parabola cutting the η-axis at the points $\lambda = 1$, $\eta = 1$ and $\lambda = -1{\cdot}431$, $\eta = 1{\cdot}9792+0{\cdot}2358\epsilon$. The success of this approximation is demonstrated by figure 8.

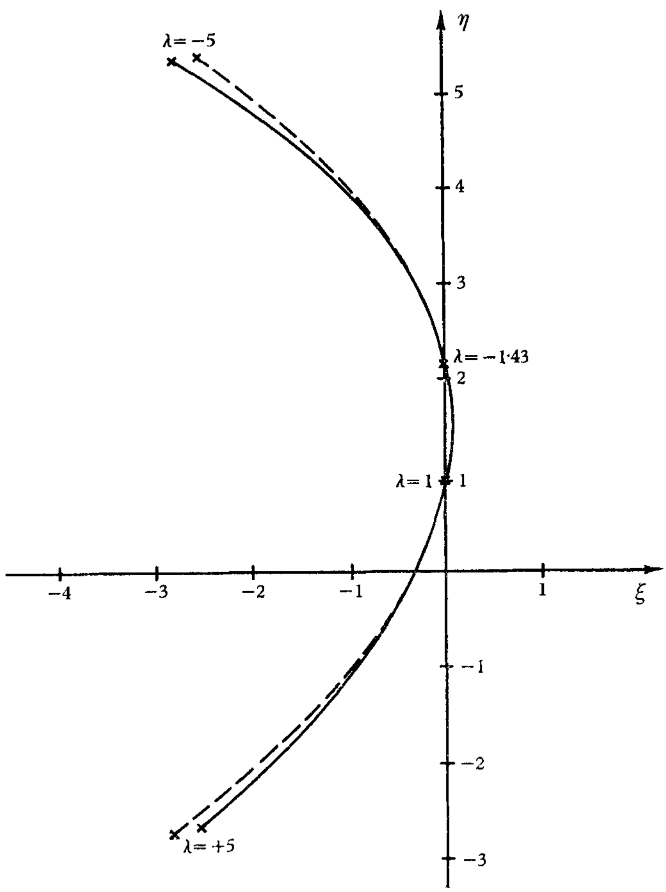

FIGURE 8. Comparison of exact and approximate curves Γ for $\epsilon = 0{\cdot}05$. ——, curve obtained by direct integration of equation (4.13); – – –, curve obtained from equation (4.16).

In this example $\psi_n(y,\lambda)$ is of the form

$$\sum_{r=0}^{n+1} \lambda^r a_r(y),$$

so that the series

$$\psi(y,\lambda) = \sum_{n=0}^{\infty} \epsilon^n \psi_n(y,\lambda)$$

may be expected to fail when $\lambda^{-1} = O(\epsilon)$. This suggests that new variables ϕ and ρ should be chosen, where

$$\phi = \lambda^{-1}\psi \quad \text{and} \quad \rho = \epsilon\lambda. \tag{4.17}$$

In terms of these variables, equation (4.13) becomes

$$\left.\begin{aligned} \phi_{yy} &= \sigma^2(1+\rho\phi)(\epsilon\rho^{-1} - y^{-1}\phi), \\ \text{with} \quad \phi(0) &= 0, \quad \phi_y(0) = 1. \end{aligned}\right\} \tag{4.18}$$

Substituting

$$\phi(y,\rho) = \sum_{n=0}^{\infty} \epsilon^n \phi_n(y,\rho), \tag{4.19}$$

$$\left.\begin{aligned} \text{we obtain} \quad \psi_{0yy} &= -\sigma^2 y^{-1}(1+\rho\phi_0)\phi_0, \\ \text{with} \quad \phi_0(0) &= 0, \quad \phi_{0y}(0) = 1, \end{aligned}\right\} \tag{4.20}$$

which is insoluble in elementary functions for $\rho \neq 0$. The equation for each ϕ_n $(n \geqslant 1)$ is linear.

The curve Γ is then given by

$$\left.\begin{aligned} \xi &= \epsilon^{-1}\rho\phi_0(1) + O(1) \\ \text{and} \quad \eta &= \epsilon^{-1}\rho\phi_{0y}(1) + O(1), \end{aligned}\right\} \tag{4.21}$$

with ρ as a variable parameter. The curve given parametrically in a (ξ_0, η_0)-plane by

$$\xi_0 = \epsilon^{-1}\rho\phi_0(1), \quad \eta_0 = \epsilon^{-1}\rho\phi_{0y}(1)$$

may be taken as a first approximation to Γ in the region where $\lambda^{-1} = O(\epsilon)$. Figure 9 shows this curve when $\sigma = \frac{1}{2}j_{1,1}$ and $\epsilon = 0{\cdot}05$, together with the curve obtained by integrating (4.13) numerically.

As $\epsilon \to 0$, for any fixed λ, $\xi(\lambda) \to 0$ and the portion of Γ between λ and λ_0 reduces to the η-axis: but if the limit $\epsilon \to 0$, $\rho_1 < \epsilon\lambda < \rho_2$ for constants $\rho_1, \rho_2 > 0$ is taken, $\xi(\lambda) \not\to 0$. In other words, Γ may be divided into an 'inner' part and an 'outer' part; the inner part tends to the η-axis, but the outer part is given by (4.21), and always is present for $\epsilon \neq 0$ at a distance from the origin which is $O(\epsilon^{-1})$.

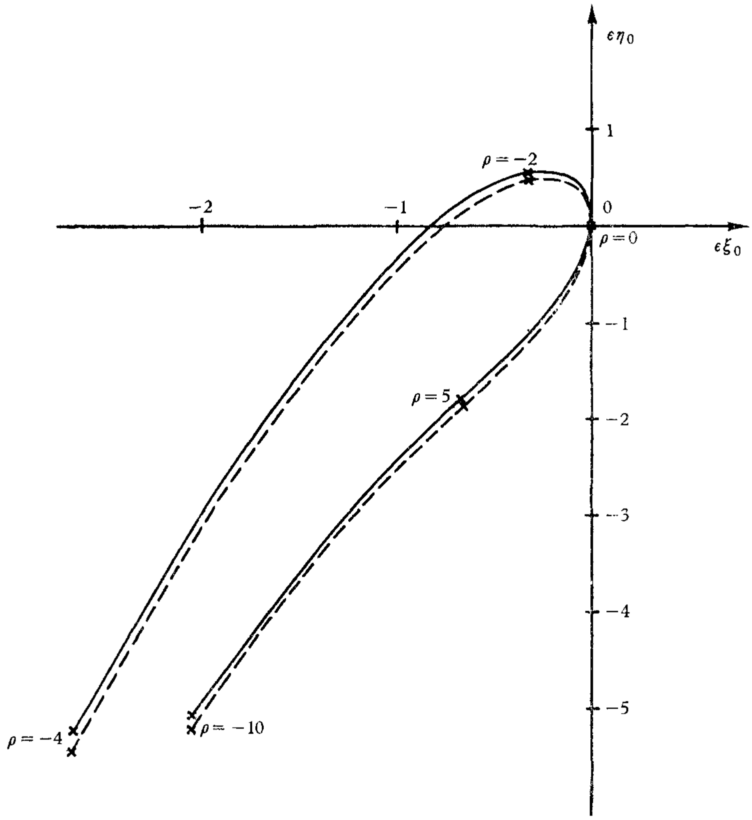

FIGURE 9. Comparison of exact and approximate curves Γ for $\epsilon = 0{\cdot}05$, $\lambda \gg 1$. ——, curve obtained by integration of equation (4.13) for large λ; – – –, curve obtained by integration of equation (4.20).

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